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LETTER TO THE EDITOR

# Metal–insulator transition in the four-dimensional Anderson model

P Markoš† and M Henneke‡

† Institute of Physics, Slovak Academy of Sciences, Dúbravaská cesta 9, 842 28 Bratislava, Slovakia

‡ PTB Braunschweig, Bundesallee 100, D-38116 Braunschweig, Germany

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**Abstract.** The metal–insulator transition in the four-dimensional Anderson model is studied numerically using the finite-size-scaling method. Critical disorder is found and the spectrum of Lyapunov exponents of the transfer matrix is shown to change from linear to the third-root shape when the system undergoes the metal–insulator transition. On the basis of our results, we conjecture the dimension dependence of the critical disorder and of the spectrum of Lyapunov exponents for dimension $d > 2$.

In this letter we present results of numerical simulations of the disorder-induced metal–insulator transition (MIT) in the four-dimensional (4D) Anderson model (AM). We find the values of the critical disorder and describe the spectrum of Lyapunov exponents (LE) of the corresponding transfer matrix in the neighbourhood of the critical point.

Besides the description of the MIT in 4D systems, there are two other motivations of the present work. (i) It would be interesting to find the connection between the analytical [1, 2] and numerical [3, 4, 5] analysis of the MIT. While the first treatment works only slightly above the critical dimension $d = 2$, numerical studies could be performed only for the smallest integer dimension ($d = 3$) for which the system exhibits MIT. Combining these results with the present work, we can conjecture the dimension dependence of the critical disorder for $d > 2$. (ii) The knowledge of the spectrum of LE is important for the studies of the statistics of LE and, consequently, of the conductance [6]. Previous numerical studies of 3D samples [7] showed that the spectrum of LE changes its form from a linear to a square-root one when the disorder $W$ crosses the critical value $W_c$. The present work enables us to generalize this result for the $d$-dimensional system ($d > 2$).

We studied the Anderson model defined by Hamiltonian

$$\mathcal{H} = W \sum_i \varepsilon_i |i\rangle\langle i| + \sum_{[NN]} |i\rangle\langle j| \tag{1}$$

where $i$, $j$ are the sites of the lattice. Only the nearest-neighbour [NN] sites enter the second sum. Random energies $\varepsilon$ have been distributed with box distribution: $|\varepsilon| < \frac{1}{2}$. $W$ measures the strength of the disorder. Analogously to the 3D case, we assume that the system is in the metallic regime for small $W$. When disorder reaches its critical value, $W_c$, the system exhibits MIT. For $W > W_c$ all states are localized.

The critical disorder $W_c$ has been found by standard methods developed for studies of the 2D and 3D systems [8, 3, 5, 10]. We consider the quasi-one-dimensional system

$$L^3 \times L_z \qquad L_z \gg L \tag{2}$$

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where $L^3$ is the three-dimensional 'cross-section' of the system. We calculate numerically the eigenvalues $\Lambda_i = \exp(2\gamma_i L_z)$ of the matrix $\mathbf{T}^\dagger\mathbf{T}$, where $\mathbf{T}$ is the transfer matrix

$$\mathbf{T} = \prod_{n=1}^{L_z} \mathbf{T}_n \qquad \mathbf{T}_n = \begin{pmatrix} E - \mathcal{H}_n & -1 \\ 1 & 0 \end{pmatrix} \tag{3}$$

and $\mathcal{H}_n$ is the Hamiltonian of the $n$th three-dimensional 'slice', perpendicular to the $z$ direction. Periodical boundary conditions have been chosen in all directions perpendicular to $z$. The length $L_z$ has been chosen to be long enough to assure the convergence of all positive LE $\gamma_i$ to their mean values with accuracy 1%.

It was supposed [8] that the MIT could be described in terms of only the first (the smallest positive) LE $\gamma_1$. For increasing $L$, the quantity

$$\Lambda(L, W) = \frac{1}{\gamma_1 L} \tag{4}$$

decreases (increases) for $W > W_c$ ($W < W_c$) [8]. The critical disorder $W_c$ can be found from the condition

$$\frac{\partial \Lambda(L, W)}{\partial L} = 0. \tag{5}$$

MacKinnon and Kramer [3] introduced the idea that $\Lambda$ can be expressed as a function of only one variable: $\Lambda(L, W) = f(L/\xi)$, where $\xi = \xi(W)$ is the scaling parameter. In the insulating regime, $\xi(W)$ represents the localization length of the $d$-dimensional model, while $\xi^{-1}$ determines the conductance of the $d$-dimensional cube for $W < W_c$. Consequently, the knowledge of the disorder dependence $\xi = \xi(W)$ in the neighbourhood of the critical point enables us to find the critical indices: $\xi \propto (W_c - W)^{-s}$ for $W < W_c$, $\xi \propto (W - W_c)^{-\nu}$ for $W > W_c$.

We calculated $\gamma_1$ for different $L$ and disorder $W$ in the neighbourhood of $W_c$. Owing to the high dimension of our problem, the size of the transfer matrix ($2L^3 \times 2L^3$) grows rapidly as $L$ increases. Therefore we had to restrict ourselves only to the system size $4 \leqslant L \leqslant 8$. Obtained data enabled us to find the critical disorder $W_c$ and to construct the scaling function $f$. However, they were unsatisfactory for determination of the critical exponents.

Figure 1 presents the $L$-dependence of $\Lambda$ and proves the existence of universal scaling function $f$. From the data we found the critical disorder

$$W_c = 33.2 \pm 0.4 \tag{6}$$

and the critical value $\Lambda_c$ of the parameter $\Lambda$

$$\Lambda_c = 0.40 \pm 0.3. \tag{7}$$

Using the data for 3D systems [4, 5]: $W_c = 16.5$, $\Lambda_c = 0.58$, we conjecture the following dimension dependences:

$$W_c(d) = W_c(d = 3)(d - 2) \tag{8}$$

$$\Lambda_c = \Lambda_c(d = 3)(d - 2)^{-1/2}. \tag{9}$$

To check relations (8) and (9) we also collect data for systems with Gaussian disorder. To save computer time, the required accuracy of $\gamma_1$ was only 3%. When using the most recent numerical value of $W_c$ in three dimensions: $W_c(d = 3) = 21.2$ [12], relation (8) predicts $W_c(d = 4) = 42.4$. On the basis of the numerical data we obtained $W_c \approx 44.7 \pm 1.1$. When taking into account the lower accuracy of the numerical data together with the small system size ($L \leqslant 8$), the agreement of the two values is satisfactory. For parameter $\Lambda_c$

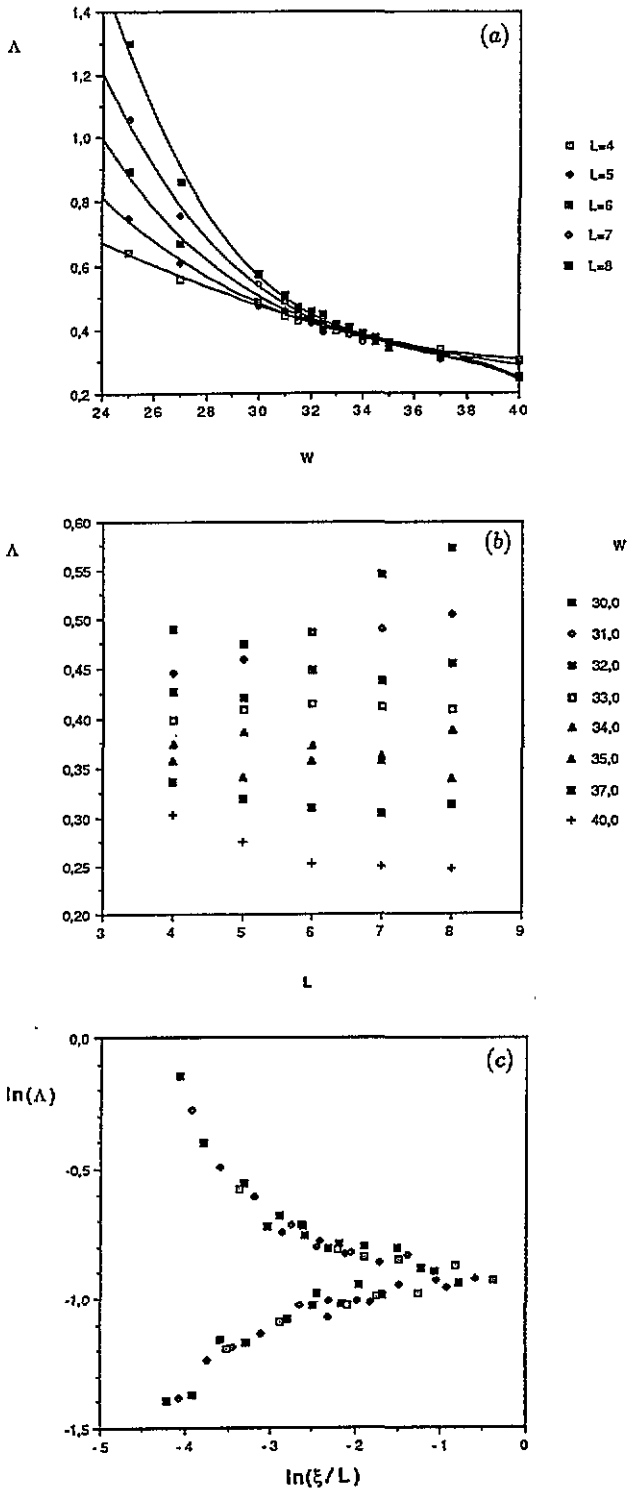


**Figure 1.** (a) Λ as a function of the disorder $W$ for different $L$. (b) Λ as a function of $L$ for different disorder (c) scaling function.

we obtained $\Lambda_c \approx 0.38 \pm 0.3$. It is, within the numerical accuracy, in agreement with (7), confirming the universality of the MIT in 4D.

For description of the spectrum of LE, it is convenient to use variables $z_i$

$$z_i = 2\gamma_i L. \tag{10}$$

Comparing with (4) we have $z_1 = 2/\Lambda$. There are $N = L^3$ positive Lyapunov exponents. However, only a small fraction of them is important for our studies. Indeed, the change of the spectrum arises only at the beginning of the spectrum. For $i \gg 1$, $z_i$ remains proportional to index $i$ in all three regimes. Nevertheless, these LE are very large. We conclude that the corresponding channels do not play any role in the MIT [11].

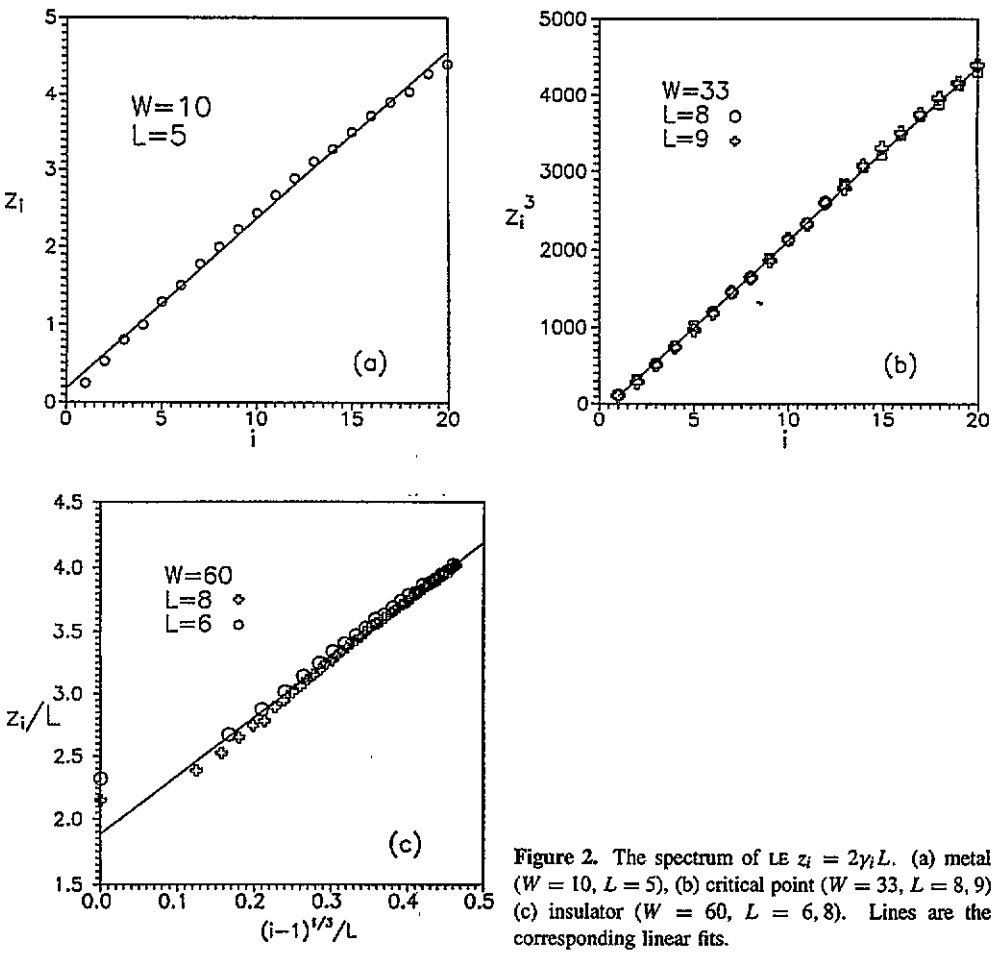


**Figure 2.** The spectrum of LE $z_i = 2\gamma_i L$. (a) metal ($W = 10$, $L = 5$), (b) critical point ($W = 33$, $L = 8, 9$) (c) insulator ($W = 60$, $L = 6, 8$). Lines are the corresponding linear fits.

The spectrum of LE for all three regimes is presented in figure 2. In the metallic regime, $W < W_c$, the linear dependence $z_i = \text{constant} \times i$ has been found (figure 2(a)). It agrees with the commonly accepted picture of the metallic regime, as first suggested by Imry [9]. At the critical point, however, the spectrum changes to the form

$$z_i = z_1(ai + b)^{1/3}. \tag{11}$$

A numerical fit through the first 20 LE and for $W = 33$, $L = 8$ gives $a \approx 1.93$, $b \approx -1.06$.

It is worth mentioning that spectrum (11) could be defined by using only the parameter $z_1$:

$$z_i = z_1\left\{1 + \tfrac{3}{8}z_1(i-1)\right\}^{1/3} \tag{12}$$

(figure 2(b)). Analogously, in the insulating regime we have

$$z_i \approx 2L_z/\xi + \left\{\tfrac{3}{8}z_1(W = W_c)(i-1)\right\}^{1/3} \tag{13}$$

(figure 2(c)). Formulae (12) and (13) emphasize the role of the first LE: it determines the form of the spectrum not only in the metallic limit, but also at and above the critical point.

Comparing relation (12) with the square-root behaviour in the 3D case [7], we suggest that the general form of the spectrum at the critical point reads

$$z_i = z_1\left\{1 + \frac{d-1}{2^{d-1}}z_1(d)(i-1)\right\}^{1/(d-1)} \tag{14}$$

where the $d$ dependence of $z_1$ is given by (9). This conjecture has been used in [11], where the general form of the probability distribution of LE of a $d$-dimensional system has been proposed.

In conclusion, we have presented numerical results of studies of the metal–insulator transition in the four-dimensional Anderson model. Using the finite-size-scaling hypothesis, we found for the first time the critical disorder of the MIT and gave arguments supporting the idea of the universality of the transition. Obtained data together with the previously known results for lower dimensions enabled us to propose the simple dimension dependence of the parameters of the transition and of the spectrum of Lyapunov exponents. In spite of the fact that the calculations have been restricted to the width $L \leqslant 8$ of the quasi-one-dimensional strips, we believe that our result provides a good starting point for analysis of the role of the dimension in the theory of the metal–insulator transition.

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